ON GENERALIZED BERNOULLI DISTRIBUTIONS AND THEIR RATIONALITY

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ABSTRACT. We interpret the Bernoulli distributions as distributions on the abelianization $G_{\mathbf{Q}}^{\mathrm{ab}}$ of the absolute Galois group of $\mathbf{Q}$. This generalizes to the case of an arbitrary number field K replacing $\mathbf{Q}$. We contruct nonabelian versions of the Bernoulli distributions on the absolute Galois group G_K. For these distributions we show a rationality property generalizing a theorem of Siegel and Klingen.

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1. Introduction

Our starting point is a sequence of distributions, which occur with three different interpretations: As Bernoulli distributions, its components are defined by the Bernoulli polynomials. As Hurwitz distributions, they are connected to special values of the Hurwitz zeta function. As standard distributions, one can get them as Fourier transforms of special values of Dirichlet L-series. Using this third interpretation, we show that the distributions are connected to abelian field extensions, and that they admit natural nonabelian generalizations to distributions on the absolute Galois group of algebraic number fields.

2. Artin L-functions

We begin by reviewing some basic properties of Artin L-functions. Let $L|K$ be a Galois extension of algebraic number fields. Consider a complex representation $\rho : G(L|K) \to \mathrm{GL}_n(\mathbf{C})$ of its Galois group. Let $\chi := \mathrm{tr} \circ \rho$ be the associated character, V the associated $\mathbf{C}[G(L|K)]$-module. Let S be a finite set of prime ideals of K. The Artin L-function associated to this representation without Euler factors associated to prime ideals in S is denoted by

$$\mathcal{L}_S(L|K, \rho, s) = \mathcal{L}_S(L|K, \chi, s) = \mathcal{L}_S(L|K, V, s).$$

If $S = \emptyset$, we allow ourselves to drop it from the notation,

$$\mathcal{L}(L|K, \rho, s) := \mathcal{L}_\emptyset(L|K, \rho, s) \text{ etc.}$$

Warning 1. Euler factors associated to infinite places are always excluded, but we do not consider infinite places as elements of S.

The following fundamental properties of Artin L-functions are shown in [4], ch. VII, prop. 10.4 for $S = \emptyset$; the proof treats individual Euler factors separately, so it is also valid for $S \neq \emptyset$. The map $\chi \mapsto \mathcal{L}_S(L|K, \chi, s)$ is a homomorphism from the additive semigroup of characters of $G(L|K)$ to the multiplicative group $\mathcal{M}(\mathbf{C})^*$

of the field of meromorphic functions on $\mathbf{C}$. It can be extended canonically to the group of all virtual characters. If $L'|K$ is another finite Galois extension such that $L \subseteq L'$, then ρ induces a representation of $G(L'|K)$. For the associated L-functions we have

$$\mathcal{L}_S(L'|K, \rho, s) = \mathcal{L}_S(L|K, \rho, s). \tag{1}$$

It would therefore be possible to exclude the field L from the notation and define the L-function $\mathcal{L}_S(K, \rho, s)$ for any representation ρ of the absolute Galois group G_K that factors over a finite quotient. Nevertheless we stick to the traditional notation and always specify a field L over which the representation under consideration is definable. Likewise, for $L|M|K$, every character ψ of $G(L|M)$ induces a character ψ_* of $G(L|K)$, and we have

$$\mathcal{L}_{\tilde{S}}(L|M, \psi, s) = \mathcal{L}_S(L|K, \psi_*, s) \tag{2}$$

for $\tilde{S} := \{\mathfrak{P} \text{ of } M : \mathfrak{P}|\mathfrak{p} \text{ for a } \mathfrak{p} \in S\}$. For the character 1 associated to the trivial representation of $G(L|K)$ and $S = \emptyset$ we get the Dedekind zeta function associated to the field K,

$$\mathcal{L}(L|K, 1, s) = \zeta_K(s).$$

3. Transformation properties of special L values under $\mathrm{Aut}(\mathbf{C})$-operation

In this section we examine how field automorphisms of $\mathbf{C}$ act on special values of Artin L-functions. (See proposition 2.) This will be crucial for the rationality of the standard distributions to be shown in theorem 1. We consider $s = 1 - k$ for $k \in \mathbf{N}$. For the particular value $s = 0$, one gets a special case of Stark's conjecture, which Tate shows in [7], ch. III, th. 1.2. The following considerations should be understood as generalizations of this theorem. For odd k, the proof is very similar to the one given by Tate, for even k one has to work differently.

For the time being let G be an arbitrary finite group. The group $\mathrm{Aut}(\mathbf{C})$ acts from the left on the set of complex valued class functions of G by

$$\chi^\alpha := \alpha \circ \chi.$$

Lemma 1. *The operation of* $\mathrm{Aut}(\mathbf{C})$ *on the set of class functions of* G *fulfills:*

(i) *The set of virtual characters and the set of characters are invariant.*
(ii) *For virtual characters, the dimension* $\chi(1)$ *is invariant.*
(iii) *For any two class functions* χ *and* ψ *we have*

$$(\chi + \psi)^\alpha = \chi^\alpha + \psi^\alpha.$$

(iv) *The set of irreducible characters is invariant.*

Proof. Consider the following operation of $\mathrm{Aut}(\mathbf{C})$ on complex G-modules: Let V be a G-module and $\alpha \in \mathrm{Aut}(\mathbf{C})$. Define $V^\alpha := V$ as set. Let the scalar multiplication be given by

$$z \cdot_\alpha x := z^{\alpha^{-1}} x := \alpha^{-1}(z)x.$$

Let elements of the group act on V^α the same way as on V, i. e. by

$$\sigma \cdot_\alpha x := \sigma x.$$

This operation is linear, because on the one hand

$$\sigma \cdot_\alpha (x_1 + x_2) = \sigma(x_1 + x_2) = \sigma x_1 + \sigma x_2 = \sigma \cdot_\alpha x_1 + \sigma \cdot_\alpha x_1$$

for $\sigma \in G$ and $x_{1/2} \in V$, and on the other hand

$$\sigma \cdot_\alpha (z \cdot_\alpha x) = \sigma(z^{\alpha^{-1}} x) = z^{\alpha^{-1}}(\sigma x) = z \cdot_\alpha (\sigma \cdot_\alpha x)$$

for $\sigma \in G,\ z \in \mathbf{C}$ and $x \in V$.

Now let $(e_1, \dots, e_n)$ be a basis of V. We show that (e_i) is also a basis of V^α: Let $x \in V$ arbitrary. Then x can be expressed as $x = \sum_i z_i e_i$ with $z_i \in \mathbf{C}$. Hence

$$x = \sum_i z_i e_i = \sum_i (z_i^\alpha)^{\alpha^{-1}} e_i = \sum_i z_i^\alpha \cdot_\alpha e_i,$$

which means that x can be expressed as a linear combination of the e_i in V^α, too. Hence (e_i) generates V^α. Let now contrarily be $z_i \in \mathbf{C}$ such that $\sum_i z_i \cdot_\alpha e_i = 0$. Then

$$0 = \sum_i z_i \cdot_\alpha e_i = \sum_i z_i^{\alpha^{-1}} \cdot e_i.$$

As (e_i) is a basis of V, it follows that $z_i^{\alpha^{-1}} = 0$ for all i. So $z_i = 0$ for all i, and (e_i) is a basis of V^α.

Let (z_{ij}) be the matrix of $\sigma \in G$ as acting on V. For all i we have

$$\sigma \cdot_\alpha e_i = \sigma e_i = \sum_j z_{ij} e_j = \sum_j (z_{ij}^\alpha)^{\alpha^{-1}} e_j = \sum_j z_{ij}^\alpha \cdot_\alpha e_j.$$

Hence the matrix of σ as acting on V^α is (z_{ij}^α). So also the traces fulfill $\mathrm{tr}_{V^\alpha} \sigma = (\mathrm{tr}_V \sigma)^\alpha$.

Hence the described operation of $\mathrm{Aut}(\mathbf{C})$ on G-modules induces the desired operation on characters. This implies the first statement. The second and third statements are obvious. The fourth statement can be seen as follows: By (i), the set of characters is invariant, by (iii) the subset of reducible characters is invariant. Hence the difference of these sets, namely the set of irreducible characters, is invariant, too. ☐

Now let H be a subgroup of G. To every class function ψ on H one can associate the induced class function ψ_* on G as follows: In a first step, extend ψ to G by $\psi(\sigma) := 0$ for $\sigma \notin H$. Then ψ_* is given by

$$\psi_*(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \psi(\tau\sigma\tau^{-1})$$

for $\sigma \in G$. The operation of $\mathrm{Aut}(\mathbf{C})$ on the set of class functions is compatible with induction:

Lemma 2. *Let H be a subgroup of the finite group G. Then for every $\alpha \in \mathrm{Aut}(\mathbf{C})$ and every class function ψ on H, we have*

$$(\psi_*)^\alpha = (\psi^\alpha)_*.$$

Proof. Using the above explicit formula for induced functions, one verifies directly:

$$\begin{aligned}\psi_*(\sigma)^\alpha &= \left(\frac{1}{|H|} \sum_{\tau \in G} \psi(\tau\sigma\tau^{-1})\right)^\alpha \\ &= \frac{1}{|H|} \sum_{\tau \in G} \psi^\alpha(\tau\sigma\tau^{-1}) \\ &= (\psi^\alpha)_*(\sigma)\end{aligned}$$

for every $\sigma \in H$. $\square$

In the sequel, let $L|K$ be a Galois extension of algebraic number fields again, $G := G(L|K)$ and S a finite set of prime ideals of K. For every G-module V and every $k \in \mathbf{N}$ let $r_{S,k}(V)$ denote the order of vanishing of the associated Artin L-function $\mathcal{L}_S(L|K, V, s)$ at $s = 1-k$. For $S = \emptyset$ we shorten the notation to $r_k(V) := r_{\emptyset,k}(V)$.

For every (finite of infinite) place $\mathfrak{P}$ of L let $\varphi_{\mathfrak{P}}$ be an associated Frobenius automorphism. For $\mathfrak{P}|\infty$, it is uniquely determined, as infinite places are always nonramified. In this case let $V^{\varphi_{\mathfrak{P}}} := \{x \in V : \rho(\varphi_{\mathfrak{P}})x = x\}$ and $V^{-\varphi_{\mathfrak{P}}} := \{x \in V : -\rho(\varphi_{\mathfrak{P}})x = x\}$. The vector space V decomposes as a direct sum $V = V^{\varphi_{\mathfrak{P}}} \oplus V^{-\varphi_{\mathfrak{P}}}$, see Neukirch, [4], ch. VII, § 12.

Lemma 3. *For every place $\mathfrak{p}$ of K choose an extension $\mathfrak{P}|\mathfrak{p}$ of L. The orders of vanishing $r_{S,k}(V)$ fulfill*

$$r_{S,k}(V) = \begin{cases} -\dim V^G + \sum_{\mathfrak{p}|\infty} \dim V^{\varphi_{\mathfrak{P}}} + \sum_{\mathfrak{p}\in S} \dim V^{G_{\mathfrak{P}}} & k = 1, \\ |\{\mathfrak{p} : \mathfrak{p} \textit{ complex}\}| \cdot \dim V + \sum_{\mathfrak{p} \textit{ real}} \dim V^{-\varphi_{\mathfrak{P}}} & k = 2, 4, 6, \dots, \\ \sum_{\mathfrak{p}|\infty} \dim V^{\varphi_{\mathfrak{P}}} & k = 3, 5, 7, \dots \end{cases}$$

In particular, $r_{S,k}(V)$ is independent of S for $k \neq 1$. Furthermore, $r_{S,k}(V)$ is nonnegative for all $k \in \mathbf{N}$.

Proof. The case $k = 1$ is shown by Tate in [7], ch. I, pr. 3.4. Please note the following: The set S of excluded Euler factors does not appear in Tate's formula explicitly, but by convention 3.0 it is implied. The infinite places are not dealt with especially, as they are always elements of S in Tate's book.

The remaining cases ($k \geq 2$) can be calculated by applying the functional equation for complete Artin L-functions, taking into account orders of vanishing of missing Euler factors, as follows:

Let $\Gamma(s)$ be the gamma function,

$$L_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s) \quad \text{and} \quad L_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma(s/2).$$

For every prime ideal $\mathfrak{p}$ of K let

$$\mathcal{L}_{\mathfrak{p}}(L|K, \chi, s) := \det(1 - \varphi_{\mathfrak{P}}(\mathfrak{N}\mathfrak{p})^{-s}|V^{I_{\mathfrak{P}}})^{-1}$$

denote the Euler factor associated to $\mathfrak{p}$. For infinite places $\mathfrak{p}$, let the Euler factors be given by

$$\mathcal{L}_{\mathfrak{p}}(L|K, \chi, s) := \begin{cases} L_{\mathbf{C}}(s)^{\dim V} & \mathfrak{p} \text{ complex}, \\ L_{\mathbf{R}}(s)^{\dim V^{\varphi_{\mathfrak{P}}}} L_{\mathbf{R}}(s+1)^{\dim V^{-\varphi_{\mathfrak{P}}}} & \mathfrak{p} \text{ real}. \end{cases}$$

Now consider the *complete Artin L-function* associated to V,

$$\Lambda(L|K, \chi, s) := c(\chi)^{s/2} \prod_{\mathfrak{p} \text{ place}} \mathcal{L}_{\mathfrak{p}}(L|K, \chi, s),$$

where $c(\chi) > 0$ is a suitable constant. (See Neukirch, [4], ch. VII, def. 12.2.) The complete L-function fulfills the functional equation

$$\Lambda(L|K, \chi, s) = W(\chi)\Lambda(L|K, \bar{\chi}, 1-s)$$

with a constant $W(\chi)$ of absolute value 1 (Neukirch, [4], ch. VII, th. 12.6). In particular, for all $s_0 \in \mathbf{C}$ the orders of vanishing in s_0 respectively $1 - s_0$ fulfill

$v_{s_0}(\Lambda(L|K,\chi,s)) = v_{1-s_0}(\Lambda(L|K,\bar{\chi},s))$. If we substitute $s_0 = 1-k$ for $k \geq 2$ in this formula, we get

$$v_{1-k}(\Lambda(L|K,\chi,s)) = v_k(\Lambda(L|K,\bar{\chi},s)) = 0.$$

The second equality in this formula is obtained as follows: As a convergent product, $\mathcal{L}(L|K,\bar{\chi},s)$ cannot have any zeros or poles in $\{\operatorname{Re} s > 1\}$. The remaining finitely many factors of the complete Artin L-function cannot have any zeros or poles there, either, as the gamma function has no zeros or poles in $\{\operatorname{Re} s > 0\}$.

So with $S_\infty := \{\mathfrak{p} \text{ of } K : \mathfrak{p}|\infty\}$ we have

$$\begin{aligned} 0 &= v_{1-k}(\Lambda(L|K,\chi,s)) \\ &= v_{1-k}\left(c(\chi)^{s/2} \cdot \mathcal{L}_S(L|K,\chi,s) \cdot \prod_{\mathfrak{p} \in S \cup S_\infty} \mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)\right) \\ &= 0 + r_{S,k}(V) + \sum_{\mathfrak{p} \in S \cup S_\infty} v_{1-k}(\mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)). \end{aligned}$$

Resolving this for $r_{S,k}(V)$ we get

$$r_{S,k}(V) = -\sum_{\mathfrak{p} \in S \cup S_\infty} v_{1-k}(\mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)). \tag{3}$$

The problem is hence reduced to determining the order of vanishing of the Euler factors appearing on the right hand side of (3) at $s = 1-k$.

To begin with, let $\mathfrak{p} \in S$. The Frobenius automorphism $\varphi_{\mathfrak{P}}$ has finite order (as an element of the finite group $G(L|K)$), so all its eigenvalues are roots of unity. In particular, all eigenvalues have absolute value 1. Hence the corresponding Euler factor $\det(1 - \varphi_{\mathfrak{P}}(\mathfrak{N}\mathfrak{p})^{-s}|V^{I_{\mathfrak{P}}})^{-1}$ cannot have a pole at $s \in \mathbf{R} \setminus \{0\}$, that means $v_{1-k}(\mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)) = 0$.

Now let $\mathfrak{p}$ be complex. Then

$$\begin{aligned} v_{1-k}(\mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)) &= v_{1-k}(L_{\mathbf{C}}(s)^{\dim V}) \\ &= \dim V \cdot v_{1-k}(L_{\mathbf{C}}(s)) \\ &= \dim V \cdot v_{1-k}(2(2\pi)^{-s}\Gamma(s)) \\ &= -\dim V. \end{aligned}$$

Finally let $\mathfrak{p}$ be real. Then

$$\begin{aligned} v_{1-k}(L_{\mathbf{R}}(s)) = v_{1-k}(\pi^{-s/2}\Gamma(s/2)) &= v_{1-k}(\Gamma(s/2)) = v_{(1-k)/2}(\Gamma(s)) \\ &= \begin{cases} -1 & (1-k)/2 \text{ integral, i.e. } k \text{ odd,} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

This implies

$$\begin{aligned} v_{1-k}(\mathcal{L}_{\mathfrak{p}}(L|K,\chi,s)) &= v_{1-k}(L_{\mathbf{R}}(s)^{\dim V^{\varphi_{\mathfrak{P}}}} L_{\mathbf{R}}(s+1)^{\dim V^{-\varphi_{\mathfrak{P}}}}) \\ &= \dim V^{\varphi_{\mathfrak{P}}} \cdot v_{1-k}(L_{\mathbf{R}}(s)) + \dim V^{-\varphi_{\mathfrak{P}}} \cdot v_{1-k}(L_{\mathbf{R}}(s+1)) \\ &= \begin{cases} -\dim V^{\varphi_{\mathfrak{P}}} & k \text{ odd,} \\ -\dim V^{-\varphi_{\mathfrak{P}}} & k \text{ even.} \end{cases} \end{aligned}$$

Substituting these orders of vanishing in formula (3) yields for even k:

$$\begin{aligned} r_{S,k}(V) &= -\sum_{\mathfrak{p}\in S\cup S_\infty} v_{1-k}(\mathcal{L}_\mathfrak{p}(L|K,\chi,s)) \\ &= \sum_{\mathfrak{p}\text{ complex}} \dim V + \sum_{\mathfrak{p}\text{ real}} \dim V^{-\varphi_\mathfrak{P}} \\ &= |\{\mathfrak{p} : \mathfrak{p}\text{ complex}\}| \cdot \dim V + \sum_{\mathfrak{p}\text{ real}} \dim V^{-\varphi_\mathfrak{P}}. \end{aligned}$$

For odd k, we get accordingly

$$\begin{aligned} r_{S,k}(V) &= -\sum_{\mathfrak{p}\in S\cup S_\infty} v_{1-k}(\mathcal{L}_\mathfrak{p}(L|K,\chi,s)) \\ &= \sum_{\mathfrak{p}\text{ complex}} \dim V + \sum_{\mathfrak{p}\text{ real}} \dim V^{\varphi_\mathfrak{P}} \\ &= \sum_{\mathfrak{p}|\infty} \dim V^{\varphi_\mathfrak{P}}. \end{aligned}$$

The last equality holds, because for complex $\mathfrak{p}$ we always have $\varphi_\mathfrak{P} = 1$ and hence $V^{\varphi_\mathfrak{P}} = V$. □

Lemma 4. *The numbers $r_{S,k}$ are invariant under the operation of* $\mathrm{Aut}(\mathbf{C})$*, i. e. for all $\alpha \in \mathrm{Aut}(\mathbf{C})$, $k \in \mathbf{N}$ we have*

$$r_{S,k}(V^\alpha) = r_{S,k}(V).$$

Proof. Given the explicit formula in lemma 3 and because G acts on V^α as it does on V, it is sufficient to show that the dimension of every complex vector space V is invariant under the operation of $\mathrm{Aut}(\mathbf{C})$, i. e.

$$\dim V = \dim V^\alpha$$

for all $\alpha \in \mathrm{Aut}(\mathbf{C})$. This is actually true, because every basis of V is a basis of V^α, too, as we have seen in the proof of lemma 1. □

For a character χ of $G(L|K)$ let L_χ denote the minimal extension of K such that χ is defined over its Galois group, i. e.

$$L_\chi = L^{\ker \rho_\chi}.$$

A totally imaginary number field is called a *CM field*, if it is a quadratic extension of a totally real number field. A number field is a CM field if and only if it has a nontrivial automorphism, uniquely (i. e. independently of the embedding into $\mathbf{C}$) induced by complex conjugation. (See Lang, [2], ch. 1, § 2.)

The field L_χ governs the vanishing behavior of $\mathcal{L}(L|K,\chi,s)$ as follows:

Proposition 1. *Let χ be a character of $G(L|K)$.*

(i) *If χ does not contain the trivial character as a summand, and if $r_{S,1}(\chi) = 0$, then K is totally real and L_χ is a CM field.*
(ii) *If $k \in \{3, 5, 7, \ldots\}$ and $r_{S,k}(\chi) = 0$, then K is totally real and L_χ a CM field.*
(iii) *For even k the following holds: $r_{S,k}(\chi) = 0$ if and only if L_χ is totally real.*

Proof. Because of the functoriality of Artin L-function in equation (1), the order of vanishing $r_{S,k}(\chi)$ does not depend on L ab, but only on χ as a character of the absolute Galois group G_K. The same holds for the minimal extension L_χ of K, over the Galois group of which χ is defined. One can hence suppose that $L = L_\chi$, i. e. that the representation is injective.

We begin with considering odd k. By the explicit calculation of the order of vanishing in lemma 3, in this case $r_{S,k}(\chi) = 0$ implies that $V^{\varphi_\mathfrak{P}} = 0$ for all $\mathfrak{p}|\infty$. The order of the automorphisms $\varphi_\mathfrak{P}$ divides 2. Hence the Jordan canonical form of $\rho(\varphi_\mathfrak{P})$ can only contain 1 and -1 on the diagonal, besides the diagonal only 0. Together with $V^{\varphi_\mathfrak{P}} = 0$ this implies that $\rho(\varphi_\mathfrak{P}) = -1$ for all $\mathfrak{p}$ and all choices of $\mathfrak{P}|\mathfrak{p}$. As the representation is injective, this implies that all $\varphi_\mathfrak{P}$ coincide. Hence K is totally real and L is a CM field.

Now consider even k. Because of lemma 3, in this case $r_{S,k}(\chi) = 0$ means that there are no complex places and that $V^{-\varphi_\mathfrak{P}} = 0$ for all real $\mathfrak{p}$. As above, this implies that $-\rho(\varphi_\mathfrak{P}) = -1$, hence $\rho(\varphi_\mathfrak{P}) = 1$, hence $\varphi_\mathfrak{P} = 1$ for all $\mathfrak{p}$ and $\mathfrak{P}$, meaning that all $\mathfrak{p}|\infty$ are totally decomposed. Hence L has to be totally real, too. If contrarily L is totally real, then K has no complex places and the real $\mathfrak{p}$ are totally decomposed in L, implying that $\varphi_\mathfrak{P} = 1$. From lemma 3 one can then read off that $r_{S,k}(\chi) = 0$. □

For the proof of propostition 2 we need the following refinement, given by Serre, of Brauer's induction theorem:

Lemma 5. *Let G be a finite group with center Z and χ an irreducible character of G. Then the restriction of χ to Z is a multiple of a one-dimensional character $\psi : Z \to \mathbf{C}^*$, and χ can be expressed as*

$$\chi = \sum_i n_i \chi_{i,*}.$$

Here, H_i are subgroups of G and contain Z; χ_i are one-dimensional characters of H_i, the restriction of which to Z is ψ; $n_i \in \mathbf{Z}$ are integers.

Proof. See Tate, [7], ch. III, le. 1.3. □

Lemma 6. *Let χ be a character of $G(L|K)$ and $k \in \mathbf{N}$ with $\mathcal{L}_S(L|K, \chi, 1-k) \neq 0$. Then there are intermediate fields K_i of $L|K$, one-dimensional characters χ_i of $G(L|K_i)$ and $n_i \in \mathbf{Z}$, such that*

(i) $\chi = \sum_i n_i \chi_{i,*}$ *and*
(ii) $\mathcal{L}_{S_i}(L|K_i, \chi_i, 1-k) = \mathcal{L}_S(L|K, \chi_{i,*}, 1-k) \neq 0$

with $S_i := \{\mathfrak{P}$ of $K_i : \mathfrak{P}|\mathfrak{p}$ for a $\mathfrak{p} \in S\}$.

Proof. We begin by supposing that χ is irreducible. If χ is one-dimensional, there is nothing to show. Therefore assume the contrary, in particular $\chi \neq 1$. Because of equation (1) about the independence of Artin L-functions of the field L, one can assume $L = L_\chi$. The first part of (ii) is simply equation (2) about Artin L-functions for induced characters, so it even holds in much greater generality.

Consider k even. Proposition 1 in this case says that L is totally real. By Brauer's (original) induction theorem, there are intermediate fields K_i of $L|K$, one-dimensional characters χ_i of $G(L|K_i)$ and $n_i \in \mathbf{Z}$, such that (i) holds. As L is totally real, by proposition 1, also the L functions $\mathcal{L}_S(L|K, \chi_{i,*}, s)$ do not vanish in $s = 1-k$. This shows the second part of (ii).

Nor consider k odd. Proposition 1 in this case says that K is totally real and that L is a CM field. Let Z be the center of $G(L|K)$. By lemma 5 we can choose intermediate fields K_i of $L|K$ with $Z \subseteq G(L|K_i)$, one-dimensional characters χ_i of $G(L|K_i)$ and $n_i \in \mathbf{Z}$. These fulfill (i). As L is a CM field, all $\varphi_{\mathfrak{P}}$ for $\mathfrak{P}|\infty$ coincide. Let τ be this automorphism. Let $j : L \to \mathbf{C}$ be an arbitrary embedding. Then τ is given by $\tau(z) = j^{-1}(\overline{j(z)})$. For arbitrary $\sigma \in G(L|K)$ we have

$$\begin{aligned}(\sigma\tau\sigma^{-1})(z) &= \sigma j^{-1}(\overline{j(\sigma^{-1}z)}) \\ &= (j\sigma^{-1})^{-1}(\overline{(j\sigma^{-1})(z)}) \\ &= \tau(z),\end{aligned}$$

because $j\sigma^{-1}$ is an embedding of L in $\mathbf{C}$, too. Hence $\tau \in Z$. Let $\psi : Z \to \mathbf{C}^*$ be as in lemma 5. In the proof to proposition 1 we have seen that $\rho(\tau) = -1$. Hence $\chi(\tau) = -\chi(1)$. As $\chi = \chi(1)\psi$ on Z, it follows that $\psi(\tau) = -1$. Because of $\chi_i = \psi$ on Z, it also follows that

$$\chi_i(\tau) = -1 \tag{4}$$

for all i. Now fix i, let $\mathfrak{p}$ be an infinite place of K_i and $\mathfrak{P}$ its extension to L. Then, the Frobenius automorphism $\varphi_{\mathfrak{P}}$ for $\mathfrak{P}$ is given by $\varphi_{\mathfrak{P}}(z) = \tilde{\jmath}^{-1}(\overline{\tilde{\jmath}(z)})$ for an embedding $\tilde{\jmath} : L \to \mathbf{C}$. Hence $\varphi_{\mathfrak{P}} = \tau$. Because of (4) we conclude that $\chi_i(\varphi_{\mathfrak{P}}) = -1$, hence $V_i^{\varphi_{\mathfrak{P}}} = 0$. By lemma 3 it follows that $r_{S_i,k}(\chi_i) = 0$. So the L-function $\mathcal{L}_{S_i}(L|K_i, \chi_i, s)$ does not vanish in $s = 1-k$. This shows (ii) for odd k, too.

Finally, we consider the case of reducible $\chi = \sum_j \chi^{(j)}$. We have supposed that $\mathcal{L}_S(L|K, \chi, 1-k) \neq 0$. Artin L-functions cannot have poles in $s = 1-k$ by lemma 3. Hence $\mathcal{L}_S(L|K, \chi^{(j)}, 1-k) \neq 0$ has to hold for all j. So by the facts already shown, for every irreducible component $\chi^{(j)}$ of χ we find a decomposition $\chi^{(j)} = \sum_i n_i^{(j)} \chi_i^{(j)}$ satisfying (i) and (ii). Then

$$\chi = \sum_j \chi^{(j)} = \sum_j \sum_i n_i^{(j)} \chi_i^{(j)}$$

is a decomposition of χ satisfying (i) and (ii). □

Proposition 2. *Let χ be a character of $G(L|K)$. Then for all $\alpha \in \mathrm{Aut}(\mathbf{C})$ and $k \in \mathbf{N}$ we have*

$$\mathcal{L}_S(L|K, \chi, 1-k)^\alpha = \mathcal{L}_S(L|K, \chi^\alpha, 1-k).$$

Proof. Once the proposition is shown for all irreducible characters, we can conclude for arbitrary $\chi = \sum_i \chi_i$ that

$$\begin{aligned}\mathcal{L}_S(L|K, \chi, s)^\alpha &= \prod_i \mathcal{L}_S(L|K, \chi_i, s)^\alpha \\ &= \prod_i \mathcal{L}_S(L|K, \chi_i^\alpha, s) \\ &= \mathcal{L}_S(L|K, \chi^\alpha, s).\end{aligned}$$

We therefore assume without loss of generality χ as irreducible in the sequel.

For an ideal $\mathfrak{m}$ of K let $J^{\mathfrak{m}}$ denote the group of ideals of K which are relatively prime to $\mathfrak{m}$, and $P^{\mathfrak{m}}$ the subgroup of principal ideals which are generated by a totally positive element in the class of 1 modulo $\mathfrak{m}$. So by class field theory, we have

$G(K^{\mathfrak{m}}|K) = J^{\mathfrak{m}}/P^{\mathfrak{m}}$, where $K^{\mathfrak{m}}$ denotes the ray class field for $\mathfrak{m}$. In the projective limit, we get $G_K^{ab} = \varprojlim J^{\mathfrak{m}}/P^{\mathfrak{m}}$.

Consider one-dimensional χ first. In this case, the Artin L-function $\mathcal{L}_S(L|K, \chi, 1-k)$ corresponds to a Hecke L-series,

$$\mathcal{L}_S(L|K, \chi, s) = L(\chi, s) = \sum_{(\mathfrak{a},\mathfrak{m})=1} \chi(\mathfrak{a})(\mathfrak{N}\mathfrak{a})^{-s}$$

for an integral ideal $\mathfrak{m}$ of K, where we have denoted the Hecke character corresponding to χ on $J^{\mathfrak{m}}/P^{\mathfrak{m}}$ by χ, too. This Hecke L-series further decomposes

$$\begin{aligned}\mathcal{L}_S(L|K, \chi, s) &= L(\chi, s) \\ &= \sum_{(\mathfrak{a},\mathfrak{m})=1} \chi(\mathfrak{a})(\mathfrak{N}\mathfrak{a})^{-s} \\ &= \sum_{\mathfrak{a} \text{ rep. } J^{\mathfrak{m}}/P^{\mathfrak{m}}} \chi(\mathfrak{a}) \sum_{\substack{\mathfrak{b}\in\mathfrak{a}P^{\mathfrak{m}} \\ \mathfrak{b} \text{ integral}}} (\mathfrak{N}\mathfrak{b})^{-s} \\ &= \sum_{\mathfrak{a} \text{ rep. } J^{\mathfrak{m}}/P^{\mathfrak{m}}} \chi(\mathfrak{a}) H(s, \mathfrak{a}P^{\mathfrak{m}})\end{aligned}$$

with the *partial zeta functions* $H(s, \mathfrak{a}P^{\mathfrak{m}}) := \sum_{\mathfrak{b}\in\mathfrak{a}P^{\mathfrak{m}},\ \mathfrak{b} \text{ integral}} (\mathfrak{N}\mathfrak{b})^{-s}$. (Confer example 4.)

The above calculation is valid for $\operatorname{Re} s > 1$. As the first and last terms have a unique meromorphic extension to $\mathbf{C}$, they coincide there, too. The values of the partial zeta functions $H(s, \mathfrak{a}P^{\mathfrak{m}})$ at $s = 1-k,\ k \in \mathbf{N}$ are rational (in particular finite) by the theorem of Siegel/Klingen, see Neukirch, [4], ch. VII, co. 9.9. Hence

$$\mathcal{L}_S(L|K, \chi, 1-k)^\alpha = \sum_{\mathfrak{a} \text{ rep. } J^{\mathfrak{m}}/P^{\mathfrak{m}}} \chi(\mathfrak{a})^\alpha H(s, \mathfrak{a}P^{\mathfrak{m}}) = \mathcal{L}_S(L|K, \chi^\alpha, 1-k),$$

as desired.

We proceed to the general case: Let χ be an arbitrary irreducible character of $G(L|K)$. In the proof of lemma 1 we have seen that one cat get the $G(L|K)$-module V^α associated to χ^α by retaining the operation of $G(L|K)$ on V and changing only the vector space structure on V. This implies that $\ker\rho = \ker\rho^\alpha$, hence that $L_\chi = L_{\chi^\alpha}$. Therefore we can suppose without loss of generality that $L = L_\chi = L_{\chi^\alpha}$, i. e. that the representations ρ and ρ^α are injective. The case $\chi = 1$ was already treated in the first step. We therefore exclude it in the sequel, as it would require some special handling for $k = 1$. Because of lemma 4, the L-function for χ vanishes at $s = 1-k$ if and only if the one for χ^α vanishes. We can therefore suppose that $r_{S,k}(\chi) = 0$. We want to reduce the general case to the case already treated by Brauer induction.

By lemma 6 we choose intermediate fields K_i of $L|K$, one-dimensional characters χ_i of $G(L|K_i)$ and $n_i \in \mathbf{Z}$ such that $\chi = \sum_i n_i\chi_{i,*}$ and that the L-series for the χ_i do not vanish at $s = 1-k$. For one-dimensional characters, we have already shown the desired transformation formula. Hence we can calculate directly:

$$\begin{aligned}\mathcal{L}_S(L|K, \chi, 1-k)^\alpha &= \mathcal{L}_S(L|K, \sum_i n_i\chi_{i,*}, 1-k)^\alpha \\ &= \left(\prod_i \mathcal{L}_S(L|K, \chi_{i,*}, 1-k)^{n_i}\right)^\alpha \qquad \text{as there is no zero}\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_i \mathcal{L}_{S_i}(L|K_i, \chi_i, 1-k)^{n_i}\right)^{\alpha} && \text{by (2)} \\
&= \prod_i \big(\mathcal{L}_{S_i}(L|K_i, \chi_i, 1-k)^{\alpha}\big)^{n_i} \\
&= \prod_i \mathcal{L}_{S_i}(L|K_i, \chi_i^{\alpha}, 1-k)^{n_i} && \text{as } \chi_i \text{ onedim.} \\
&= \prod_i \mathcal{L}_{S}(L|K, (\chi_i^{\alpha})_*, 1-k)^{n_i} && \text{by (2)} \\
&= \prod_i \mathcal{L}_{S}(L|K, (\chi_{i,*})^{\alpha}, 1-k)^{n_i} && \text{by le. 2} \\
&= \mathcal{L}_S(L|K, \sum_i n_i (\chi_{i,*})^{\alpha}, 1-k) \\
&= \mathcal{L}_S(L|K, \left(\sum_i n_i \chi_{i,*}\right)^{\alpha}, 1-k) \\
&= \mathcal{L}_S(L|K, \chi^{\alpha}, 1-k).
\end{aligned}
$$

□

4. Distributions on profinite spaces

Definition 1. Let $X = \varprojlim X_i$ $(i \in I)$ be a profinite topological space and A an abelian group. A map $F : \bigcup_i X_i \to A$ is called a *distribution* on X with values in A, if for all $i, j \in I,\ i \leq j$ and $x \in X_i$ the distribution relation

$$F(x) = \sum_{\substack{y \in X_j \\ y \mapsto x}} F(y) \tag{5}$$

is satisfied. We denote the abelian group of distributions on X with values in A by $\operatorname{Dist}(X, A)$.

Remark 1. The distribution relation (5) states that

$$\operatorname{Dist}(X, A) = \varprojlim \operatorname{Fun}(X_i, A),$$

if we choose the maps φ_{ij} in the projective system $\big(\operatorname{Fun}(X_i, A)\big)_{i \in I}$ as

$$(\varphi_{ij} F)(x) := \sum_{y \mapsto x} F(y).$$

Example 1. For $N, a \in \mathbf{N}$ let

$$H(s, a \bmod N) := \sum_{\substack{n \geq 1 \\ n \equiv a\ (N)}} n^{-s}.$$

Then $(a \bmod N) \mapsto H(s, a \bmod N)$ is a distribution on $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/N\mathbf{Z}$ with values in the complex vector space of formal Dirichlet series in a variable s. The Dirichlet series $H(s, a \bmod N)$ converge for $\operatorname{Re} s > 1$ and have meromorphic continuations on $\mathbf{C}$ with a single pole at $s = 1$ (see Washington, [8], ch. 4). So we can also regard these maps as distributions with values in the field $\mathcal{M}(\mathbf{C})$ of meromorphic functions on $\mathbf{C}$.

Example 2. Let $B_k(x)$ be the k-th Bernoulli polynomial, given by the generating series

$$\frac{ze^{xz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}.$$

For $x \in \mathbf{R}$ let $\langle x \rangle$ denote the element of $x + \mathbf{Z}$ such that $0 < \langle x \rangle \leq 1$. Then

$$E_k(a \bmod N) := N^{k-1}\frac{B_k(\langle a/N \rangle)}{k}$$

defines a distribution E_k on $\hat{\mathbf{Z}}$ with values in $\mathbf{Q}$. (This is shown e.g. in Lang, [3], ch. 2, § 2.) E_k is called the die k-th *Bernoulli distribution.*

Warning 2. A common definition of $\langle x \rangle$ treats the boundary cases conversely. For our purposes, the given definition is more suitable, as otherwise proposition 4 would not hold for $k = 1$.

Lemma 7. *Let F be a distribution on X with values in A and $\varphi : A \to B$ a homomorphism of abelian groups. Then $\varphi \circ F$ is a distribution on X with values in B.*

Proof. Clear. □

Example 3. Let $s_0 \in \mathbf{C},\ s_0 \neq 1$. With H as in example 1, the map

$$(a \bmod m) \mapsto H(s_0, a \bmod m)$$

defines a distribution on $\hat{\mathbf{Z}}$ with values in $\mathbf{C}$. The homomorphism φ is in this case evaluation at s_0. The distributions obtained this way for $s_0 = 1-k,\ k \in \mathbf{N}$ is called the k-th *Hurwitz distribution* H_k.

Distributions with values in a module can be used for integration of functions: Let $X = \varprojlim X_i$ be a profinite topological space as before, R a commutative ring and M an R-module. For $i \in I$ let $\pi_i : X \to X_i$ be the canonical projection. Then $\pi_i^{-1}(\{y\})$ is an open closed subset of X for $y \in X_i$.

A function $f : X \to R$ is said to *factor over* X_i, if there is a map $\tilde{f} : X_i \to R$ such that $f = \tilde{f} \circ \pi_i$. A function $f : X \to R$ is said to *factor over an* X_i, if there is an i such that f factors over X_i. Let $\mathcal{S}(X, R)$ denote the R-module of all functions which factor over an X_i. These are precisely the locally constant functions.

Definition 2. Let F be a distribution on X with values in M and $f \in \mathcal{S}(X, R)$. Then

$$\int_X f(y)F(dy) := \sum_{y \in X_i} \tilde{f}(y)F(y)$$

is called the *integral* of f over X with respect to F, where i is chosen sufficiently large so that f factors over X_i.

For sets $A \subseteq B$ let $1_A : B \to \{0, 1\}$ denote the characteristic function of A.

Proposition 3. *The R-modules* $\mathrm{Dist}(X, M)$ *and* $\mathrm{Hom}_R(\mathcal{S}(X, R), M)$ *are canonically isomorphic. The distribution F corresponds to the homomorphism Φ with*

$$\Phi(f) = \int_X f(y)F(dy).$$

Conversely, to a homomorphism Φ corresponds the distribution F with

$$F(y) := \Phi(1_{\pi_i^{-1}(\{y\})})$$

for $y \in X_i$.

Proof. We show that both compositions of the given maps are the identity. First, let $F \in \mathrm{Dist}(X, M)$. Let Φ be the homomorphism with $\Phi(f) = \int_X f(y)F(dy)$. Then, for every $i \in I$ and $y \in X_i$ we have

$$\begin{aligned}\Phi(1_{\pi_i^{-1}(\{y\})}) &= \int_X 1_{\pi_i^{-1}(\{y\})}(z)F(dz)\\ &= \sum_{z\in X_i} \tilde{1}_{\pi_i^{-1}(\{y\})}(z)F(z)\\ &= \sum_{z\in X_i} 1_{\{y\}}(z)F(z)\\ &= F(y).\end{aligned}$$

Conversely, let $\Phi \in \mathrm{Hom}_R(\mathcal{S}(X, R), M)$. Let F be the distribution with $F(y) := \Phi(1_{\pi_i^{-1}(\{y\})})$ for $y \in X_i$. Then, for every $i \in I$ and every $f \in \mathcal{S}(X, R)$ that factors over an X_i we have

$$\begin{aligned}\int_X f(y)F(dy) &= \sum_{y\in X_i} \tilde{f}(y)F(y)\\ &= \sum_{y\in X_i} \tilde{f}(y)\Phi(1_{\pi_i^{-1}(\{y\})})\\ &= \Phi\left(\sum_{y\in X_i} \tilde{f}(y)1_{\pi_i^{-1}(\{y\})}\right)\\ &= \Phi(f).\end{aligned}$$

□

Occasionally we will identify both sides of the isomorphism. Therefore we impose

$$F(f) := \int_X f(y)F(dy).$$

Remark 2. Proposition 3 adjusts a looseness in our definition of a distribution: We talk about distributions on the profinite topological space X, but the definition depends on the chosen representation $X = \varprojlim X_i$. One should therefore talk about distributions on the projective system (X_i). Let A be an abelian group. By proposition 3 (for $R = \mathbf{Z},\ M = A$) the group $\mathrm{Dist}(X, A)$ is canonically isomorphic to $\mathrm{Hom}_{\mathbf{Z}}(\mathcal{S}(X, \mathbf{Z}), A)$. Hence $\mathrm{Dist}(X, A)$ is indeed an invariant of the topological space X.

5. Bernoulli and Hurwitz distributions

This section compiles some properties of the Bernoulli distributions E_k from example 2 and the Hurwitz distributions H_k from example 3.

Proposition 4. *For all $k \in \mathbf{N}$,*

$$H_k = -E_k,$$

i. e. the Bernoulli and Hurwitz distributions agree up to sign. In particular, H_k has values in $\mathbf{Q}$, too.

Proof. For $0 < x \leq 1$ let $\zeta(s,x) := \sum_{i=0}^{\infty}(x+i)^{-s}$ denote the Hurwitz zeta function, meromorphically extended on $\mathbf{C}$. For $N \in \mathbf{N},\ 1 \leq a \leq N$ we have

$$\begin{aligned} H(s, a \bmod N) &= \sum_{k=0}^{\infty}(a+kN)^{-s} \\ &= N^{-s}\sum_{k=0}^{\infty}(a/N+k)^{-s} \\ &= N^{-s}\zeta(s,a/N) \end{aligned}$$

The above calculation is valid for $\operatorname{Re} s > 1$. As the first and the last terms have unique meromorphic extensions on $\mathbf{C}$, they agree there, too. By Washington, [8], th. 4.2, we have $\zeta(1-k,x) = -B_k(x)/k$ for $0 < x \leq 1$, altogether

$$\begin{aligned} H_k(a \bmod N) &= H(1-k, a \bmod N) \\ &= N^{k-1}\zeta(1-k,a/N) \\ &= -N^{k-1}B_k(a/N)/k \\ &= -E_k(a \bmod N). \end{aligned}$$

The fact that we have chosen a as the representative in $\{1,\ldots,N\}$ corresponds to the condition $0 < \langle x\rangle \leq 1$, cf. the warning on page 11. □

Example 4. The Hurwitz distributions from example 1 and 3 can be regarded as distributions on $\mathbf{Z}_p$ and $\mathbf{Z}_p^*$ for any prime number p by restriction. The version defined on $\mathbf{Z}_p^*$ admits the following generalization: Let K be an algebraic number field. Recall that for an ideal $\mathfrak{m}$ of K we let $J^{\mathfrak{m}}$ denote the group of ideals of K which are relatively prime to $\mathfrak{m}$, and $P^{\mathfrak{m}}$ the subgroup of principal ideals which are generated by a totally positive element in the class of 1 modulo $\mathfrak{m}$. (So by class field theory, we have $G(K^{\mathfrak{m}}|K) = J^{\mathfrak{m}}/P^{\mathfrak{m}}$, where $K^{\mathfrak{m}}$ denotes the ray class field for $\mathfrak{m}$. In the projective limit, we get $G_K^{ab} = \varprojlim J^{\mathfrak{m}}/P^{\mathfrak{m}}$.) For any ideal $\mathfrak{a}$ of K relatively prime to $\mathfrak{m}$, let

$$H(s,\mathfrak{a}P^{\mathfrak{m}}) := \sum_{\substack{\mathfrak{b}\in\mathfrak{a}P^{\mathfrak{m}} \\ \mathfrak{b}\text{ integral}}} (\mathfrak{N}\mathfrak{b})^{-s}$$

be the *partial zeta function* associated to $\mathfrak{a}$, which we already used in the proof of proposition 2. Now fix a prime ideal $\mathfrak{p}$ of K and let $K^{\mathfrak{p}^\infty} := \bigcup_{n\in\mathbf{N}} K^{\mathfrak{p}^n}$ be the union of all ray class fields associated to powers of $\mathfrak{p}$. If we consider $\mathfrak{m}$ of the form $\mathfrak{m} = \mathfrak{p}^n$, we get a distribution on $\varprojlim J^{\mathfrak{p}}/P^{\mathfrak{p}^n} = \varprojlim J^{\mathfrak{p}^n}/P^{\mathfrak{p}^n} = G(K^{\mathfrak{p}^\infty}|K)$ with values in the vector space of formal Dirichlet series in s or (after meromorphic extension) with values in the field of meromorphic functions on C. These functions do not have poles at $s = 1-k$ for $k \in \mathbf{N}$. (Indeed they take rational values there by the theorem of Siegel/Klingen cited below.) Hence by lemma 7 we get distributions H_k^K with values in $\mathbf{C}$ for $k \in \mathbf{N}$ by

$$H_k^K(\mathfrak{a}P^{\mathfrak{p}^n}) := H(1-k,\mathfrak{a}P^{\mathfrak{p}^n}).$$

We refer to these distributions as *Hurwitz distributions*, too.

Warning 3. The choice of a fixed prime ideal $\mathfrak{p}$ seems superfluous at first sight: For any class $\mathfrak{a}P^{\mathfrak{m}} \in J^{\mathfrak{m}}/P^{\mathfrak{m}}$ one can define $H(s,\mathfrak{a}P^{\mathfrak{m}})$. However, these function do not fulfill the distribution relation (5), so one does not get a distribution on $\varprojlim J^{\mathfrak{m}}/P^{\mathfrak{m}} = G_K^{ab}$. This is because of missing Euler factors in Hecke L-series associated to nonprimitive characters. By using Artin L-functions one can repair this defect (see section 8).

Proposition 5 (Siegel, Klingen)**.** *The Hurwitz distributions for every algebraic number field K take values in $\mathbf{Q}$, i. e.*

$$H_k^K \in \operatorname{Dist}(\varprojlim J^{\mathfrak{p}}/P^{\mathfrak{p}^n}, \mathbf{Q}).$$

Proof. See e. g. Neukirch, [4], ch. VII, co. 9.9. □

6. The algebra of distributions

Let $G = \varprojlim G_i$ $(i \in I)$ be a profinite group and A a ring. The abelian group $\operatorname{Dist}(G, A)$ then becomes an A-module, if A acts value-wise on distributions.

Definition 3. Define the *convolution product* on $\operatorname{Dist}(G, A)$ by

$$(F * G)(c) := \sum_{ab=c} F(a)G(b)$$

for $c \in G_i$, where (a, b) runs through the pairs of elements of G_i which fulfill $ab = c$. With the convolution product, $\operatorname{Dist}(G, A)$ becomes an A-algebra.

Definition 4. For a profinite group $G = \varprojlim G_i$ $(i \in I)$ and a ring A let

$$A[[G]] := \varprojlim A[G_i]$$

be the *complete group ring* of G with coefficients in A.

Proposition 6. *The A-algebras* $\operatorname{Dist}(G, A)$ *and* $A[[G]]$ *are canonically isomorphic.*

Proof. By the isomorphism $\operatorname{Dist}(G, A) = \varprojlim \operatorname{Fun}(G_i, A)$ from the remark on page 10, the two objects are clearly isomorphic as abelian groups and A-modules. We are left to satisfy ourselves that the products coincide. Indeed, the product in the group rings $A[G_i]$ is defined the same way as the convolution product. □

Definition 5. For an element $\theta = \sum c_\sigma \cdot \sigma$ in an arbitrary group ring call $\bar{\theta} := \sum c_\sigma \cdot \sigma^{-1}$ the element *opposite* to θ.

Example 5. Let $F \in \operatorname{Dist}(\mathbf{Z}_p^*, \mathbf{Q})$ be the restriction of the first Bernoulli distribution E_1 to $\mathbf{Z}_p^* = G(\mathbf{Q}(\mu_{p^\infty})|\mathbf{Q}) = \varprojlim G(\mathbf{Q}(\mu_{p^n})|\mathbf{Q})$. For $a \in (\mathbf{Z}/p^n\mathbf{Z})^*$ let $\sigma_a \in G(\mathbf{Q}(\mu_{p^n})|\mathbf{Q})$ be the automorphism which acts on p^n-th roots of unity by $\zeta^{\sigma_a} = \zeta^a$. Because of $B_1(x) = x - 1/2$, the distribution F corresponds as an element of $\mathbf{Q}[[G(\mathbf{Q}(\mu_{p^\infty})|\mathbf{Q})]]$ to the family $(\bar{\theta}(p^n))_{n\in\mathbf{N}}$ with

$$\begin{aligned}\bar{\theta}(p^n) &= \sum_{\substack{a=1,\ldots,p^n-1\\(a,p)=1}} B_1(a/p^n) \cdot \sigma_a \\ &= \sum_{\substack{a=1,\ldots,p^n-1\\(a,p)=1}} \left(\frac{a}{p^n} - \frac{1}{2}\right) \cdot \sigma_a.\end{aligned}$$

Hence one gets by $\bar{\theta}(p^n)$ opposite Stickelberger elements. Generally, by considering E_k $(k \in \mathbf{N})$ one gets the opposite k-th Stickelberger elements $\theta_k(p^n)$ from Lang, [3], ch. 2, § 3.

(Rather would one consider all N instead of p^n. This is not possible because of the reasons given in the warning about example 4: The restriction of E_1 to $\hat{\mathbf{Z}}^*$ is not a distribution.)

7. Class distributions and scalar Fourier transformation

On profinite groups one gets distributions as Fourier transforms of character functions (see proposition 7). In order to show this, we need the following orthogonality relation:

Lemma 8. *Let G be a finite group, H normal in G, furthermore χ an irreducible character and c an element of G. Then*

$$\sum_{\substack{\tilde{c}\in G\\ \tilde{c}H=cH}} \chi(\tilde{c}) = \begin{cases} |H|\cdot\chi(\tilde{c}) & \text{if } \rho_\chi \text{ factors over } G/H,\\ 0 & \text{else,}\end{cases}$$

where ρ_χ denotes a representation corresponding to χ.

Proof. The first case is obvious. In the second case, for all irreducible characters χ' such that $\rho_{\chi'}$ factors over G/H we have $\chi \neq \chi'$. Hence by the orthogonality relations it follows that

$$(\chi,\chi')_G := \frac{1}{|G|}\sum_{a\in G}\chi(a)\bar{\chi}'(a) = 0.$$

Let $f : G/H \to \mathbf{C}$ be given by

$$f(aH) := \sum_{b\in H}\chi(ab).$$

f is a class function on G/H, because

$$\begin{aligned} f(xax^{-1}H) &= \sum_{b\in H}\chi(xax^{-1}b)\\ &= \sum_{b\in H}\chi(xax^{-1}xbx^{-1})\\ &= \sum_{b\in H}\chi(xabx^{-1})\\ &= \sum_{b\in H}\chi(ab).\end{aligned}$$

For every irreducible character χ' of G/H, we have

$$\begin{aligned} (f,\chi')_{G/H} &= \frac{1}{(G:H)}\sum_{a \bmod H} f(a)\bar{\chi}'(a)\\ &= \frac{1}{(G:H)}\sum_{a \bmod H}\sum_{b\in H}\chi(ab)\bar{\chi}'(ab)\\ &= \text{const.}\cdot(\chi,\chi')_G\\ &= 0.\end{aligned}$$

As the χ' form an orthonormal basis of the space of complex valued class functions on G/H, we conclude that $f = 0$. □

Let $G = \varprojlim G_i$ $(i \in I)$ be a profinite group and V a complex vector space.

Definition 6. A *character function* of G with values in V is a map which associates an element of V to every irreducible character of G factorizing over a G_i. Let

ChFct(G, V) denote the complex vector space of character functions on G with values in V.

A distribution F on a profinite group G with values in V is called a *class distribution*, if it is constant on every conjugacy class of every finite quotient G_i. Let ClDist(G, V) denote the set of class distributions on G with values in V.

Proposition 7. *The vector spaces* ClDist(G, V) *and* ChFct(G, V) *are canonically isomorphic. To a distribution F corresponds the character function Φ with*

$$\Phi(\chi) = \int_G \chi(\sigma) F(d\sigma).$$

Conversely, to a character function Φ corresponds the die distribution F with

$$F(\sigma) = \frac{1}{|G_i|} \sum_{\chi \text{ of } G_i} \Phi(\chi) \cdot \bar{\chi}(\sigma)$$

for $\sigma \in G_i$.

Proof. The crucial point is to show that the association $\Phi \mapsto F$ indeed always yields a distribution. One has to verify the distribution relation (5). So let $i \leq j,\ \sigma \in G_i$. Then

$$\begin{aligned}
\sum_{\substack{\tau \in G_j \\ \tau \mapsto \sigma}} F(\tau) &= \frac{1}{|G_j|} \sum_{\chi \text{ of } G_j} \Phi(\chi) \sum_{\tau} \bar{\chi}(\tau) \\
&= \frac{1}{|G_i|} \sum_{\chi \text{ of } G_i} \Phi(\chi) \bar{\chi}(\sigma) \\
&= F(\sigma),
\end{aligned}$$

where the second equality is justified by the fact that $\sum_\tau \bar{\chi}(\tau)$ is 0 or $\frac{|G_j|}{|G_i|}\bar{\chi}(\sigma)$ by lemma 8. F is defined as a linear combination of characters. Hence F is constant on conjugacy classes, hence a class distribution.

The fact that the two maps are inverse to each other can now be verified immediately using orthogonality relations for characters as follows: First, let Φ be an arbitrary character function and F the distribution accordingly defined as in the proposition. Then for every character χ which factors over G_i we have

$$\begin{aligned}
\int_G \chi(\sigma) F(d\sigma) &= \sum_{\sigma \in G_i} \chi(\sigma) F(\sigma) \\
&= \sum_{\sigma \in G_i} \chi(\sigma) \frac{1}{|G_i|} \sum_{\psi \text{ of } G_i} \Phi(\psi) \bar{\psi}(\sigma) \\
&= \sum_{\psi \text{ of } G_i} \Phi(\psi) \frac{1}{|G_i|} \sum_{\sigma \in G_i} \chi(\sigma) \bar{\psi}(\sigma) \\
&= \sum_{\psi \text{ of } G_i} \Phi(\psi) (\chi, \psi) \\
&= \Phi(\chi).
\end{aligned}$$

Conversely, let F be an arbitrary class distribution and Φ the character function accordingly defined as in the proposition. Then for $\sigma \in G_i$ we have

$$\begin{aligned}
\frac{1}{|G_i|} \sum_{\chi \text{ of } G_i} \Phi(\chi)\bar{\chi}(\sigma) &= \frac{1}{|G_i|} \sum_{\chi \text{ of } G_i} \int_G \chi(\tau) F(d\tau) \bar{\chi}(\sigma) \\
&= \frac{1}{|G_i|} \sum_{\chi \text{ of } G_i} \sum_{\tau \in G_i} \chi(\tau) F(\tau) \bar{\chi}(\sigma) \\
&= \sum_{\chi \text{ of } G_i} \frac{1}{|G_i|} \left(\sum_{\tau \in G_i} F(\tau)\chi(\tau) \right) \bar{\chi}(\sigma) \\
&= \sum_{\chi \text{ of } G_i} (F|_{G_i}, \bar{\chi}) \bar{\chi}(\sigma) \\
&= \sum_{\chi \text{ of } G_i} (F|_{G_i}, \chi) \chi(\sigma) \\
&= \left(\sum_{\chi \text{ of } G_i} (F|_{G_i}, \chi) \chi \right) (\sigma) \\
&= F(\sigma).
\end{aligned}$$

□

By virtue of proposition 7 we will occasionally identify distributions and corresponding character functions and denote them with the same letter.

The distributions obtained by propositioin 7 are constant on every conjugacy class of every finite quotient of the profinite group under consideration. Therefore it is more easy to work directly with distributions on a suitable class space. This is possible as follows:

Proposition 8. *Let $G = \varprojlim G_i$ $(i \in I)$ be a profinite group. For every finite quotient G_i let $\mathrm{Cl}(G_i)$ the set of conjugacy classes. Let F be a distribution on G which is constant on every element of every $\mathrm{Cl}(G_i)$. For $c \in \mathrm{Cl}(G_i)$, $\sigma \in c$ let*

$$\underline{F}(c) := |c|\, F(\sigma) = \sum_{\tau \in c} F(\tau).$$

This defines a distribution $\underline{F}$ on the profinite topological space $\mathrm{Cl}(G) := \varprojlim \mathrm{Cl}(G_i)$.

Proof. The projective limit $\varprojlim \mathrm{Cl}(G_i)$ is well defined, as conjugate elements in a finite quotients G_j are mapped to conjugate elements of the smaller quotient G_i $(i \leq j)$. It remains to verify the distribution relation (5). To this end, let $i \leq j$, $c \in \mathrm{Cl}(G_i)$. Then

$$\begin{aligned}
\underline{F}(c) &= \sum_{\tau \in c} F(\tau) \\
&= \sum_{\tau \in c} \sum_{\tilde{\tau} \mapsto \tau} F(\tilde{\tau}) \\
&= \sum_{\tilde{c} \mapsto c} \sum_{\tilde{\tau} \in \tilde{c}} F(\tilde{\tau}) \\
&= \sum_{\tilde{c} \mapsto c} \underline{F}(\tilde{c}),
\end{aligned}$$

Where the passage to the third line is justified by the following argument: In the first case we sum over $\{\tilde{\tau} \in G_j : \varphi_{ij}(\tilde{\tau}) \sim \sigma\}$, in the second case over $\{\tilde{\tau} \in G_j : \tilde{\tau} \sim \tilde{\tau}' \text{ and } \varphi_{ij}(\tilde{\tau}') \sim \sigma \text{ for a } \tilde{\tau}' \in G_j\}$. In fact, these sets coincide. □

8. The standard distributions

Proposition 7 allows us to transform character functions to distributions. We take advantage of this fact for the following construction.

Let K be an algebraic number field, G_K the absolute Galois group of K and S a finite set of prime ideals of K. For every character χ of every finite quotient $G(L|K)$ let $\mathcal{L}_S(L|K, \rho_\chi, s)$ denote the corresponding Artin L-function without Euler factors associated to prime ideals in S. For $L|K$ finite Galois and $\sigma \in G(L|K)$ let

$$L_S(s, \sigma) := \frac{1}{|G(L|K)|} \sum_{\chi \text{ of } G(L|K)} \mathcal{L}_S(L|K, \rho_\chi, s) \cdot \bar{\chi}(\sigma)$$

be the *partial zeta function* for σ. Then $\sigma \mapsto L_S(s, \sigma)$ is a distribution on G_K with values in $\mathcal{M}(\mathbf{C})$ by proposition 7. The term "'partial zeta function"' justified by the fact that $L_S(s, 1) = \zeta_{K,S}(s)$ for the trivial automorphism $1 \in G_{\mathbf{Q}}$, where $\zeta_{K,S}(s)$ denotes the Dedekind zeta function for K without Euler factors associated to prime ideals in S. Now the distribution relation (5) states that every finite Galois extension $L|K$ entails an additive decomposition

$$\zeta_{K,S}(s) = \sum_{\sigma \in G(L|K)} L_S(s, \sigma)$$

of the Dedekind zeta function.

As in example 3 one can evaluate the partial zeta functions at points s_0 where none of them has a pole. This gives rise to a distribution $\sigma \mapsto L_S(s_0, \sigma)$ with values in $\mathbf{C}$. By lemma 3, Artin L-functions do not have poles in $s = 1 - k$. Hence partial zeta functions cannot have poles there, either. So the construction indicated above is possible for $s_0 = 1 - k$.

Definition 7. Denote the distribution obtained this way by $L_{S,k}^K$, i. e.

$$L_{S,k}^K(\sigma) = L_S(1 - k, \sigma).$$

$L_{S,k}^K$ is called the k-th *standard distribution* for S on G_K. For $K = \mathbf{Q}$ we abbreviate to $L_{S,k} := L_{S,k}^{\mathbf{Q}}$.

Proposition 7 in this case attests that

$$\mathcal{L}_S(L|K, \chi, 1 - k) = \int_{G_K} \chi(\sigma) L_{S,k}^K(d\sigma) \tag{6}$$

for all irreducible characters χ of every finite quotient of G_K.

Warning 4. Equation (6) becomes false if we drop the condition of irreducibility of χ: This is already clear from the fact that the left hand side behaves multiplicatively under addition of characters, while the right hand side behaves additively.

The functions $L_S(s, \sigma)$ are connected with the functions $H(s, a \bmod m)$ from example 1 as follows:

Proposition 9. *Let p be a prime number, $n \in \mathbf{N}$. Let $\sigma \in G(\mathbf{Q}(\mu_{p^n})|\mathbf{Q})$ act on p^n-th roots of unity by $\zeta^\sigma = \zeta^a$ for $a \in (\mathbf{Z}/p^n\mathbf{Z})^*$. Then (with $S = \emptyset$)*

$$L_\emptyset(s,\sigma) = H(s, a \bmod p^n) + \frac{1}{(p-1)p^{n-1}} H(s, p \bmod p).$$

Proof. Let $G = G(\mathbf{Q}(\mu_{p^n})|\mathbf{Q})$. For every character χ of G let χ denote the corresponding *primitive* Dirichlet character, too. For $s \in \mathbf{C}$ with $\operatorname{Re} s > 1$ we have

$$\begin{aligned} |G| \cdot L(s,\sigma) &= \sum_{\chi \in \hat{G}} \mathcal{L}(\mathbf{Q}(\mu_{p^n})|\mathbf{Q}, \chi, s) \cdot \bar{\chi}(\sigma) \\ &= \sum_{\chi \in \hat{G}} \sum_{m=1}^{\infty} \chi(m) m^{-s} \chi(a^{-1}) \\ &= \sum_{\chi \in \hat{G}} \sum_{m=1}^{\infty} \chi(a^{-1}m) m^{-s} \\ &= \sum_{m=1}^{\infty} \left(\sum_{\chi \in \hat{G}} \chi(a^{-1}m) \right) m^{-s}. \end{aligned}$$

Because of

$$\sum_{\chi \in \hat{G}} \chi(a^{-1}m) = \begin{cases} 0, & \text{if } a^{-1}m \not\equiv 1 \ (p^n) \text{ and } p \nmid m, \\ |G|, & \text{if } a^{-1}m \equiv 1 \ (p^n), \text{ and} \\ 1, & \text{if } p \mid m \text{ (regard } \chi = 1) \end{cases}$$

it follows that

$$|G| \cdot L(s,\sigma) = |G| \cdot \left(\sum_{m \equiv a \ (p^n)} m^{-s} + \frac{1}{|G|} \sum_{p|m} m^{-s} \right)$$

and hence the proposition. ∎

Corollary 1. *Let p be a prime number, $n \in \mathbf{N}$. Let $\sigma \in G(\mathbf{Q}(\mu_{p^n})|\mathbf{Q})$ act on p^n-th roots of unity by $\zeta^\sigma = \zeta^a$ for $a \in (\mathbf{Z}/p^n\mathbf{Z})^*$. Then (with $S = \{p\}$)*

$$L_{\{p\}}(s,\sigma) = H(s, a \bmod p^n).$$

Proof. As $H(s, p \bmod p) = p^{-s}\zeta(s)$, proposition 9 implies

$$\begin{aligned} H(s, a \bmod p^n) &= L(s,\sigma) - \frac{1}{|G|} p^{-s} \zeta(s) \\ &= L(s,\sigma) - \frac{1}{|G|} \big(1 - (1 - p^{-s})\big) \zeta(s) \\ &= L(s,\sigma) - \frac{1}{|G|} \big(\zeta(s) - (1 - p^{-s})\zeta(s)\big) \\ &= \frac{1}{|G|} \left(\sum_{\chi \text{ of } G} \mathcal{L}(\mathbf{Q}(\mu_{p^n})|\mathbf{Q}, \chi, s) \bar{\chi}(\sigma) - \zeta(s) + (1 - p^{-s})\zeta(s) \right). \end{aligned}$$

So in the sum over all χ, the summand for the trivial character $\chi = 1$ is replaced by the corresponding summand without the p Euler factor. As $\chi(p) = 0$ for all $\chi \neq 1$

we can write

$$\begin{aligned} H(s, a \bmod p^n) &= \frac{1}{|G|}\left(\sum_{\chi \text{ of } G}(1-\chi(p)p^{-s})\mathcal{L}(\mathbf{Q}(\mu_{p^n})|\mathbf{Q},\chi,s)\bar{\chi}(\sigma)\right) \\ &= \frac{1}{|G|}\left(\sum_{\chi \text{ of } G}\mathcal{L}_{\{p\}}(\mathbf{Q}(\mu_{p^n})|\mathbf{Q},\chi,s)\bar{\chi}(\sigma)\right) \\ &= L_{\{p\}}(s,\sigma) \end{aligned}$$

and obtain the corollary. □

The connection between Hurwitz distributions and standard distributions can be seen over other ground fields than $\mathbf{Q}$, too. In this case we directly show the version corresponding to corollary 1, which is more natural anyway.

Proposition 10. *Let K be an algebraic number field, $\mathfrak{p}$ a prime ideal of K and $n \in \mathbf{N}$. Let $\mathfrak{m} := \mathfrak{p}^n$ and $L := K^{\mathfrak{m}}$ the ray class field for $\mathfrak{m}$. Let $\mathfrak{a}$ be an integral ideal of K relatively prime to $\mathfrak{p}$, and $\sigma := (\mathfrak{a}, L|K) \in G(L|K)$ the automorphism corresponding to it by the Artin symbol. Then*

$$L_{\{\mathfrak{p}\}}(s,\sigma) = H(s, \mathfrak{a}P^{\mathfrak{m}}).$$

Proof. Let $G := G(L|K)$. For a character χ of G let χ denote the (not necessarily primitive) corresponding Hecke character on the ray class group $J^{\mathfrak{m}}/P^{\mathfrak{m}}$, too. Hence $\mathcal{L}_{\{\mathfrak{p}\}}(L|K,\chi,s) = L(\chi,s)$ for every χ, where $L(\chi,s)$ denotes the Hecke L-series. We have

$$\begin{aligned} |G|\, L_{\{\mathfrak{p}\}}(s,\sigma) &= \sum_{\chi \text{ of } G}\mathcal{L}_{\{\mathfrak{p}\}}(L|K,\chi,s)\bar{\chi}(\sigma) \\ &= \sum_{\chi} L(\chi,s)\chi(\sigma^{-1}) \\ &= \sum_{\chi}\sum_{(\mathfrak{b},\mathfrak{p})=1}\chi(\mathfrak{b})(\mathfrak{N}\mathfrak{b})^{-s}\chi(\mathfrak{a}^{-1}) \\ &= \sum_{(\mathfrak{b},\mathfrak{p})=1}\left(\sum_{\chi}\chi(\mathfrak{b}\mathfrak{a}^{-1})\right)(\mathfrak{N}\mathfrak{b})^{-s}. \end{aligned}$$

Because of

$$\sum_{\chi}\chi(\mathfrak{b}\mathfrak{a}^{-1}) = \begin{cases} |G| & \mathfrak{b}\mathfrak{a}^{-1} = 1 \text{ in } J^{\mathfrak{m}}/P^{\mathfrak{m}}, \\ 0 & \text{else} \end{cases}$$

if follows that

$$|G|\, L_{\{\mathfrak{p}\}}(s,\sigma) = |G| \sum_{\substack{\mathfrak{b}\in\mathfrak{a}P^{\mathfrak{m}} \\ \mathfrak{b} \text{ integral}}}(\mathfrak{N}\mathfrak{b})^{-s} = |G|\, H(s,\mathfrak{a}P^{\mathfrak{m}}),$$

and hence the proposition. □

9. Rationality of the standard distributions

Theorem 1. *Let $L|K$ be a finite Galois extension of algebraic number fields, $\sigma \in G(L|K)$ and S an arbitrary finite set of prime ideals of K. Then*

$$L_S(1-k,\sigma) \in \mathbf{Q}$$

for the partial zeta function $L_S(s,\sigma)$ from section 8 and every $k \in \mathbf{N}$. In other words, the standard distribution $L^K_{S,k}$ on G_K has values in $\mathbf{Q}$.

Proof. Let $\alpha \in \mathrm{Aut}(\mathbf{C})$ be arbitrary. Then by proposition 2

$$\begin{aligned} L_S(1-k,\sigma)^\alpha &= \left(\frac{1}{|G|}\sum_{\chi \text{ of } G} \mathcal{L}_S(L|K,\chi,1-k)\bar{\chi}(\sigma)\right)^\alpha \\ &= \frac{1}{|G|}\sum_{\chi \text{ of } G} \mathcal{L}_S(L|K,\chi,1-k)^\alpha \bar{\chi}^\alpha(\sigma) \\ &= \frac{1}{|G|}\sum_{\chi \text{ of } G} \mathcal{L}_S(L|K,\chi^\alpha,1-k)\chi^\alpha(\sigma^{-1}) \\ &= \frac{1}{|G|}\sum_{\chi \text{ of } G} \mathcal{L}_S(L|K,\chi,1-k)\chi(\sigma^{-1}) \\ &= \frac{1}{|G|}\sum_{\chi \text{ of } G} \mathcal{L}_S(L|K,\chi,1-k)\bar{\chi}(\sigma) \\ &= L_S(1-k,\sigma). \end{aligned}$$

Hence $L_S(1-k,\sigma)$ is invariant under $\mathrm{Aut}(\mathbf{C})$ and hence rational. □

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